Appendix B

Spectral Decomposition of Diffracted Light

This section derives an expression for the relationship between the light diffracted by a holographic fringe pattern and the fringe spectrum. In keeping with the philosophy of the diffraction-specific approach, the derivation is backwards: it begins with the desired image and works backward to the fringe spectrum. First the diffracted light is decomposed into a summation of plane-wave components[16] at the plane of the hologram. Each of these plane waves is diffracted by a particular spatial frequency component of the fringe pattern.

The goal of this derivation is to determine what light must be diffracted by a fringe to generate a specified 3-D image $u(\boldsymbol{r})$. To derive the light distribution at the plane of the hologram (at $z=0$), first consider the 3-D Fourier transform[16] of this image:

$$U(\boldsymbol{k}) = \int\int\int_{-\infty}^{\infty} u(\boldsymbol{r}) e^{-i\boldsymbol{k} \cdot \boldsymbol{r}} d\boldsymbol{r} \tag{B1}$$

where

$$\boldsymbol{r} \equiv x + y + z$$

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$$u(\boldsymbol{r}) = \kappa^3 \int\int\int_{-\infty}^{\infty} U(\boldsymbol{k}) e^{i\boldsymbol{k} \cdot \boldsymbol{r}} d\boldsymbol{k}$$

$$\kappa \equiv (2\pi)^{-1} \tag{B3}$$

Thus, a 3-D object can be represented as a summation of spatial frequency components. Each of these components is a plane wave in space with a propagation vector of **k**. Plane waves are the eigenmodes of the free-space propagation of light[13]. Therefore, since the image is represented as a summation of plane waves, there is no need to apply the laws of diffraction to determine the light distribution at a particular plane. The requirements for propagation are

$$k_x^2 + k_y^2 + k_z^2 = k_0 \equiv 2\pi/\lambda$$

$$k_0^2 > k_x^2 + k_y^2 \tag{B4}$$

for propagation in the positive z direction. (The free-space wavelength of light is λ.) Combining the last two equations, the light distribution that must be diffracted by the fringe at the hologram plane $z=0$ is

$$u_D(x, y) = \kappa^2 \int_{-\infty}^{\infty}\int U_O(k_x, k_y) e^{ixk_x} e^{iyk_y} e^{-iz_O\sqrt{k_0^2 - k_x^2 - k_y^2}} dk_x dk_y \tag{B5}$$

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where $u_O(x,y)$ is the specified image in a plane at a distance $z=z_O$ from the hologram plane and $U_O(k_x, k_y)$ is the 2-D Fourier transform of the specified image[38].

This dissertation generally discusses horizontal-parallax-only (HPO) holograms. Because a single HPO holograms (hololine) diffracts light only in the x and z direction, the above expressions can be simplified to eliminate y dependence:

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(For full-parallax holography, the y dependence can be carried throughout the remaining analysis.) Note that this equation is correct for points at all depths: $z_p<0$, $z_p>0$, $z_p=0$. In general, image points or elements lie in front of, behind, and on the hologram plane.

Equation B7 states that a weighted sum of plane waves compose the diffracted distribution. The weighting factors are the Fourier transform of u_O. An additional phase term is included in each plane-wave component. Notice that this phase factor is not itself a function of x. For a given x location, the superposition (in Equation B7) used to construct u_O uses the same phase factor for each plane-wave component. To see this more clearly, consider the example of a point image at (x_p,z_p). The Fourier transform of a spatial impulse is

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In this case, the "weights" are uniform, leaving only phase factors in the superposition of plane waves. The role of the phase term is to add a fixed phase shift for each plane

wave. It represents a second order component of the diffracted wavefront, namely a curvature that is a function of z_p.

Consider the role of the phase term in Equation B10. From the point of view of a single sample of the fringe (at a single x location), each spectral component of the diffracted light must include a particular (x-independent) phase shift. And for a given hogel, each spectral component needs to include such a phase shift. As implied by the dependence of the phase term on z_p, a particular spatial frequency component of the hogel must include different phase factors for contributions to image elements at different depths. Diffraction-specific computation ignores this phase term, lumping all of the diffractive function of a particular spatial frequency into a single basis fringe. This is equivalent to fixing the wavefront curvature to zero independent of z_p and therefore reduces the image resolution. However, the basis fringes are designed to diffract light into a range of directions, not just into a single plane wave. Statistically, the effect of the phase term is small since k_x is not deterministic as is independent of z_p. Furthermore, the phase term becomes less important if spatially incoherent light is used. Equations B7 and B8 assume spatially coherent monochromatic illumination, and the superposition involves summing the complex amplitudes of plane waves. However, in the case where incoherent light[22] is used, the intensities of the plane-wave components add. Therefore, the phase term is no longer important in the relationship between fringe spectrum and light diffraction.

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$$k_x = k_x^i + 2\pi f \tag{B11}$$

where k_x is the x-component of the directional vector $\mathbf{k}$ of the (first-order) diffracted light. The point image example (Equation B10) becomes:

$$u_p(x) = \int_0^{0.5p} e^{i2\pi f(x - x_p)} e^{i\Phi(f)} df \tag{B12}$$

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In this case, the “weights” are uniform, leaving only phase factors in the superposition of plane waves. The role of the phase term is to add a fixed phase shift for each plane

wave. It represents a second order component of the diffracted wavefront, namely a curvature that is a function of z_p.

Consider the role of the phase term in Equation B10. From the point of view of a single sample of the fringe (at a single x location), each spectral component of the diffracted light must include a particular (x-independent) phase shift. And for a given hogel, each spectral component needs to include such a phase shift. As implied by the dependence of the phase term on z_p, a particular spatial frequency component of the hogel must include different phase factors for contributions to image elements at different depths. Diffraction-specific computation ignores this phase term, lumping all of the diffractive function of a particular spatial frequency into a single basis fringe. This is equivalent to fixing the wavefront curvature to zero independent of z_p and therefore reduces the image resolution. However, the basis fringes are designed to diffract light into a range of directions, not just into a single plane wave. Statistically, the effect of the phase term is small since k_x is not deterministic as is independent of z_p. Furthermore, the phase term becomes less important if spatially incoherent light is used. Equations B7 and B8 assume spatially coherent monochromatic illumination, and the superposition involves summing the complex amplitudes of plane waves. However, in the case where incoherent light[22] is used, the intensities of the plane-wave components add. Therefore, the phase term is no longer important in the relationship between fringe spectrum and light diffraction.

So far this analysis has shown that an image can be constructed by diffracting plane waves. The remaining task is to relate a particular spatial frequency component f to each diffracted plane wave. The direction of diffraction k_x is a function of the direction in which light is incident upon the fringe pattern. In optical holography, this means the illumination beam. In holovideo, it means the effective direction of light incident on the modulator in the display system. (This parameter was empirically derived by measuring the directions in which light is imaged by fringes consisting of constant spatial frequencies.) When light incident at the hologram with direction $\boldsymbol{k}^i = (k^i_x, k^i_y, k^i_y)$ is

modulated by a fringe with spatial frequency component f, the boundary condition of phase-matching[15] requires that

$$k_x = k_x^i + 2\pi f \tag{B11}$$

where k_x is the x-component of the directional vector $\mathbf{k}$ of the (first-order) diffracted light. The point image example (Equation B10) becomes:

$$u_p(x) = \int_0^{0.5p} e^{i2\pi f(x - x_p)} e^{i\Phi(f)} df \tag{B12}$$

where p is the fringe sampling pitch (in samples/mm) and $\Phi(f)$ is the phase term represented as simply a function of f. Finally, Equation B10 shows that a superposition of spatial frequencies can be used to construct a fringe that diffracts light from a given x location to form a point image. Relating k_x to geometric optics, $k_x = k_0 \sin\Theta_O$ combined with the phase-matching criterion (Equation B11) gives the useful expression

$$f = \frac{\sin\Theta_O}{\lambda} - \frac{k_x^i}{2\pi} \tag{B13}$$

which is essentially the grating equation. This expression was used in most of this thesis research for determining which spatial frequency must be used to diffract light in a particular direction.

As a final note on the linear superposition of fringe patterns, it is assumed in this thesis research that the modulation technique used by the display system to diffract light is linear. Amplitude modulation of light allows for linearity since it simply scales the wavefront by a real scalar between the values of 0.0 and 1.0. Phase modulation is also approximately linear for a modulation depth of $\Delta\phi<\pi/4$. The MIT holovideo display system using a weak phase modulation, so the condition $\Delta\phi<\pi/4$ is satisfied.